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Mathematical model of secession of the region

Abstract: The paper proposes a new general nonlinear mathematical model, which describes the process of the possibility of secession of a particular region from a certain state. The model is described by the Cauchy problem for a nonlinear two-dimensional dynamic system. The model assumes that only two categories of citizens live in a particular region of a state: the first category, which is a supporter of the center (unionists) and opposes the secession of the region; the second is a supporter of the secession of the region (secessionists, separatists), i.e. its separation from the center, with the aim of forming a new independent state. The general model implies the presence of both federal and external sides, which influence separatists and unionists respectively with various factors in order to change their opinion (will). Natural conditions are proposed under which a secession of the region is considered possible (for example, the presence of a majority or a qualified majority of citizens of the region supporting separatism). In a particular case, the absence of stakeholders external to the region, in the case of constant model coefficients, with opposite signs of demographic factors of the sides, the problem actually reduces to the predator-victim model and is described by a nonlinear two-dimensional dynamic system of the type "Lotka-Volterra." In this case, conditions were found for the coefficients of attracting opponents to the allies, demographic factors and initial conditions under which secession of the region is possible.

Keywords: mathematical model, dynamic system, unionists, separatists, secession.

The study of a number of social processes, such as assimilation of languages (peoples), globalization, settlement of political conflicts, secession of regions, territorial integrity of states, etc. is of great interest.

From our point of view, the only scientific approach to an adequate quantitative and qualitative description of these problems is the mathematical modeling of processes, i.e. the creation of mathematical models describing these current problems.

We have previously proposed original mathematical models: linguistic globalization, which establishes, within the framework of the model, the possibility of globalization in English [1]; two and three levels of assimilation of languages (peoples) by more common languages [2, 3].

We also proposed mathematical models for the settlement of political (not military confrontation) conflicts through the economic cooperation of parts of the populations of the sides with the participation of international organizations and relevant investment funds [4-6].

Over the past few decades, due to the desire of some political players to redistribute the world map, issues related to the self-determination of nations and the possibility of creating independent states have become relevant.

As you know, according to the principles of international law, there are two fundamental principles: the inviolability of the borders of states recognized by the UN and the right of peoples to self-determination until the creation of an independent state. The two principles actually contradict each other and their interpretation is often subjective.

It seems interesting to us, from a scientific point of view, to describe by a mathematical model the dynamics of two parts of the population of a region of an independent state that pursue opposite political goals. Moreover, the first part of the population supports the territorial integrity of the existing and UN-recognized state (unionists), and the second part advocates the separation (secession) of this region from the state and the creation of a new independent state (separatists, secessionists).

We propose a new mathematical model, which is described by the following nonlinear dynamic system

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1(t)u(t) + \big(\beta_1(t) - \beta_2(t)\big)u(t)v(t) - \gamma_1(t)u(t) + \gamma_2(t)v(t) \\ \dfrac{dv(t)}{dt} = \alpha_2(t)v(t) + \big(\beta_2(t) - \beta_1(t)\big)u(t)v(t) + \gamma_3(t)u(t) - \gamma_4(t)v(t) \end{cases} \quad (1)$$

with initial conditions

$$u(0) = u_0, \quad v(0) = v_0, \quad (2)$$

where

$u(t)$ is the number of supporters of the center (unionists) in the region at time t,

$v(t)$ is the number of opponents of the center (separatists) at time t,

$\alpha_1(t), \alpha_2(t) -$ are demographic factors of the corresponding parts of the population of the region,
$\beta_1(t), \beta_2(t)$- ratios of attraction of opponents into allies,
$\gamma_1(t), \gamma_3(t)$ - coefficients of influence external for the region and the state of the side on unionists, for the purpose of their attraction on the side of separatists,
$\gamma_2(t), \gamma_4(t)$ - the coefficient of influence of the federal side (the central government of the state) on the separatists, in order to attract them to the unionist side.

In the model, it is more logical (non-triviality of the model) to assume that at the initial moment of time unionists exceed separatists $(u_0 > v_0)$.

We proposed two (weak and strong requirements) conditions under which secession of the region is possible, which implies the fulfillment of inequalities

$$\frac{v(t)}{u(t)+v(t)} > 0{,}5, \quad t > t_* \tag{3}$$

or

$$\frac{v(t)}{u(t)+v(t)} > \frac{2}{3}, \quad t > t_{**}. \tag{4}$$

A weak condition (3) implies that more than half of the population of the region supports the idea of separatism, and a strong condition (4) - more than two thirds of the population of the region (a qualified majority) supports the idea of secession of the region and the creation of a new independent state.

Consider a special case when there is no influence of forces external to the region (outside the state, as well as the federal center), and unionists and separatists only among themselves decide on the choice of the path of political development of the region.

In this case, in the system of equations (1), it is necessary to assume

$$\gamma_1(t) \equiv 0, \qquad \gamma_3(t) \equiv 0, \qquad \gamma_2(t) \equiv 0, \qquad \gamma_4(t) \equiv 0. \tag{5}$$

If (5) is executed, the nonlinear system of differential equations (1) will be rewritten as follows:

$$\begin{cases} \frac{du(t)}{dt} = \alpha_1(t)u(t) + \big(\beta_1(t) - \beta_2(t)\big)u(t)v(t) \\ \frac{dv(t)}{dt} = \alpha_2(t)v(t) + \big(\beta_2(t) - \beta_1(t)\big)u(t)v(t) \end{cases}. \tag{6}$$

Now let's consider a special case of constancy of all coefficients of the model

$$\begin{matrix} \alpha_1(t) = \alpha_1 = const, & \alpha_2(t) = \alpha_2 = const, \\ \beta_1(t) = \beta_1 = const, & \beta_2(t) = \beta_2 = const. \end{matrix} \tag{7}$$

In case (7), the system of equations (6) and initial conditions (2) takes the form

$$\begin{cases} \frac{du(t)}{dt} = \alpha_1 u(t) + (\beta_1 - \beta_2)u(t)v(t) \\ \frac{dv(t)}{dt} = \alpha_2 v(t) + (\beta_2 - \beta_1)u(t)v(t) \end{cases}, \tag{8}$$

$$u(0) = u_0, \ \ v(0) = v_0.$$

Consider several particular cases of the Cauchy problem (8).

1. $\alpha_2 = 0, \ \ \alpha_1 = 0, \ \ \beta_2 = \beta_1$.

The demographic factors of the sides are zero, and the coefficients of attracting opponents to the allies are equal among themselves. In this case, the exact solution of the system (8) has the form

$$u(t) = u_0, \ \ v(t) = v_0. \tag{9}$$

From (3), (4), (9) it is easy to determine the conditions for the possibility of secession of the region in a weak

$$v_0 > u_0$$

and strong condition

$$v_0 > 2u_0,$$

which is impossible due to the assumption (logic) of non-triviality of the model.

2. $\alpha_2 \neq 0, \ \ \alpha_1 \neq 0, \ \ \beta_2 = \beta_1$.

The demographic factors of the sides are unequal to zero, and the coefficients of attracting opponents to the allies are equal. In this case, the exact solution of the system (8) has the form

$$u(t) = u_0 e^{\alpha_1 t}, \ \ v(t) = v_0 e^{\alpha_2 t}. \tag{10}$$

Then according to (3), (10) the weak condition of secession possibility has the form

$$e^{(\alpha_2 - \alpha_1)t} > \frac{u_0}{v_0} > 1 \tag{11}$$

a strong condition

$$e^{(\alpha_2 - \alpha_1)t} > 2\frac{u_0}{v_0} > 2. \tag{12}$$

From (11), (12) it follows that when the demographic factor of separatists is less than the demographic factor of unionists, inequalities (11), (12) do not have a solution and secession of the region is impossible. If the demographic factor of separatists is greater than the demographic factor of unionists, then the solutions to inequality (11), (12) have the form

$$t > t_* = \frac{ln\frac{u_0}{v_0}}{\alpha_2-\alpha_1}, \quad t > t_{**} = \frac{ln2\frac{u_0}{v_0}}{\alpha_2-\alpha_1}. \tag{13}$$

According to (13), secession of the region is possible with a weak requirement starting from the moment of time t_*, and with a strong requirement after the moment of time t_{**}.

3. $\alpha_2 > 0,\ \alpha_1 < 0,\ \ \beta_2 < \beta_1$.

4. $\alpha_2 < 0,\ \alpha_1 > 0,\ \ \beta_2 > \beta_1$.

The third and fourth cases suggest the opposite of signs as demographic factors and differences in the coefficients of attraction of opponents into allies. Moreover, in the third and fourth cases, we have the classic predator-victim model (Lotka-Volterra system of equations), in the third case, the role of predators is played by unionists, and separatists are the victim and vice versa in the fourth case, i.e. separatists are predators, and unionists are the victim.

It is easy to obtain the first integral of the system of equations (8)

$$\alpha_2 ln\frac{u}{u_0} - (\beta_1-\beta_2)(u(t)-u_0) = \alpha_1 ln\frac{v}{v_0} + (\beta_1-\beta_2)(v(t)-v_0), \tag{14}$$

which is a closed integral curve in the first quarter of the phase plane $(O, v(t), u(t))$ of the solutions of the system of equations (8).

Analysis (14) in the third and fourth cases leads to the following inequalities for the desired functions $v(t), u(t)$

$$v_{min} \le v(t) \le v_{max}, \tag{15}$$

where v_{min}, v_{max} is the smallest and largest positive roots of the next transcendent equation

$$\alpha_2 ln\frac{a}{u_0} - (\beta_1-\beta_2)(a-u_0) = \alpha_1 ln\frac{v}{v_0} + (\beta_1-\beta_2)(v(t)-v_0), \tag{16}$$

$$a = \frac{\alpha_2}{\beta_1-\beta_2} > 0,$$

$$u_{min} \le u(t) \le u_{max}, \tag{17}$$

where u_{min}, u_{max} the smallest and largest positive roots of the next transcendent equation

$$\alpha_2 ln\frac{u}{u_0} - (\beta_1-\beta_2)(u(t)-u_0) = \alpha_1 ln\frac{b}{v_0} + (\beta_1-\beta_2)(b-v_0), \tag{18}$$

$$b = \frac{-\alpha_1}{\beta_1 - \beta_2} > 0.$$

Thus, on a closed integral curve (14) completely located in the first quarter of the phase plane $\big(O, v(t), u(t)\big)$of the solutions of the system of equations (8), it may be minimum time point $M\big(v(t_1), u(t_1)\big)$, for which weak $(v(t_1) > u(t_1))$ or strong $(v(t_1) > 2u(t_1))$ conditions of region secession possibility are fulfilled. If such a point lies on an integral curve, then the secession of the region is possible and impossible otherwise.

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